J. Fluid Mech. (1972), *vol.* 54, *part* 2, *pp.* 339–349
Printed in Great Britain

On the response of a sphere to an acoustic pulse

By SAMUEL TEMKIN
Department of Mechanical Engineering, Rutgers University, New Brunswick

(Received 13 October 1971)

The motion of a rigid sphere responding to the passage of an acoustic pulse is considered by means of a simple approximate model which neglects the diffraction of the pulse front. The model is based on a solution of the equivalent inviscid problem and assumes that, initially, the flow field around the sphere corresponds to the steady incompressible flow of an inviscid fluid over a sphere at rest. For $t > 0$, the motion is studied by means of the unsteady Stokes equations. Results for the sphere's velocity, displacement and drag are obtained in closed form in terms of tabulated functions and compared with results obtained by using the Stokes drag. It is found that, when the ratio of gas density to sphere material density is finite, the initial response of the sphere differs considerably from that predicted by the use of the Stokes drag. However, when the ratio of gas density to sphere material density is infinitesimal, the differences disappear. These results may be of some importance in the study of shock-induced droplet collisions in aerosol clouds.

1. Introduction

We consider a small rigid sphere initially at rest in a viscous gas and later set into motion by the passage of a very weak shock wave. The motion is of interest because of the possibility that, when a weak shock wave is propagated in a liquid-droplet aerosol cloud, collisions between different sized droplets may be induced as a result of their different response to the waves. This possibility has been mentioned with regard to droplet coalescence in thunderstorms (Goyer 1965*a*, *b*; Temkin 1969) and in rocket nozzles (Crowe & Willoughby 1966; Marble 1967), and is currently under experimental investigation (Temkin 1970; Yun 1970).

In order to compute droplet trajectories required to study the above possibility analytically, it is necessary to know at all times the viscous drag acting on the droplets. This information is lacking and the droplets' response to weak shock waves is sometimes calculated by means of Stokes's law, or by means of experimental correlations for the drag (Hoenig 1957). Stokes's law is probably more adequate when the shocks are very weak, but the transient character of the problem precludes its use to investigate the initial motion of the droplets. This initial response is, however, a main factor in determining whether a collision between a given droplet pair will occur after the passage of the wave. The reason for this is that the response of different sized droplets is determined, in part, by their inertia, and initially inertia effects are dominant.

In the present work, we make use of a simple model to study the motion described above, assuming that droplet deformation does not occur. The model considers an acoustic pulse, propagating in an unbounded viscous non-heat-conducting gas initially at rest and reaching a freely suspended sphere of radius R at time $t = 0$. The gas velocity u_0 behind the pulse front is assumed uniform and small compared with the pulse speed a. The sphere is assumed to be small and rigid and the ratio δ of gas density ρ to sphere material density is supposed small. Now, the pulse front passes over the sphere in a time of order R/a. During this time viscous effects develop around the sphere, but are limited to a layer with a thickness of order $(\nu R/a)^{\frac{1}{2}}$, which is small compared with R for certain values of R, a and the gas coefficient of kinematic viscosity ν. Under these conditions, the initial flow over the sphere can be studied by means of inviscid equations in terms of incident and scattered wave potentials. This is done in the appendix, where it is shown that, as is expected on physical grounds, the inviscid flow over the sphere becomes steady in a time of order R/a, and that this steady flow corresponds to the inviscid incompressible flow over a movable sphere. Furthermore, in that time the sphere has acquired a velocity $U_p \sim \frac{3}{2}\delta u_0$ which is much smaller than u_0 because of our assumption about δ. We adopt an approximate model based on these results to study the motion of the sphere. In this model it is assumed that at time $t = 0$ the inviscid flow around the sphere is fully developed, and that the sphere is still at rest. For $t > 0$ the motion is studied by means of the linearized equations of motion of a viscous incompressible fluid. The sphere's drag can then, in principle, be obtained by solving Basset's integro-differential equation (Basset 1888; Pearcey & Hill 1956; Yih 1969). However, in the present case, it is simpler to derive the solution from the unsteady Stokes equations using the Laplace transformation. We use this approach to obtain the sphere's velocity, displacement and drag. Our results are given in closed form in terms of tabulated functions. Numerical comparison of these results with those obtained using the Stokes drag shows that significant differences occur, as expected, only for small non-dimensional times, although mathematically both solutions behave quite differently for $t \rightarrow \infty$. Further, the differences decrease as δ is made smaller and disappear in the limit $\delta \rightarrow 0$.

2. Analysis

2.1. *Basic equations*

We now consider the Stokes flow of a viscous incompressible fluid over a movable sphere. The motion is referred to axes moving with the sphere, so that if we non-dimensionalize velocities with u_0, distances with R, pressure with $\mu u_0/R$ and time with R^2/ν the equations of motion are

$$\nabla . \mathbf{u} = 0, \tag{1}$$

$$\partial \mathbf{u}/\partial t + \nabla p = \nabla^2 \mathbf{u}, \tag{2}$$

and

$$\mathbf{u} = \mathbf{U}_p \quad \text{on} \quad r = 1 \quad \text{for} \quad t \geqslant 0,$$

where μ is the gas coefficient of viscosity. We take the main flow to be along the x axis, so that the motion is symmetric about the axis. The fluid velocity has then only radial and tangential components u_r and u_θ, respectively, which for

$t \geq 0$ satisfy the conditions

$$u_r = U_p \cos\theta, \quad u_\theta = -U_p \sin\theta \quad \text{on} \quad r = 1, \tag{3}$$

where θ is the polar angle. At large distances from the sphere the flow is uniform and parallel to the x axis. Consequently

$$\mathbf{u} \to \mathbf{e}_1 \quad \text{for} \quad r \to \infty,$$

where $\mathbf{e}_1$ is a unit vector along the x axis. Now, at $t = 0$, the steady flow around the sphere is inviscid and irrotational. The radial and tangential velocity components are then given by (Milne-Thompson 1950, p. 464)

$$u_r(r, \theta, 0) = (1 - 1/r^3)\cos\theta \tag{4}$$

and

$$u_\theta(r, \theta, 0) = -(1 + 1/2r^3)\sin\theta. \tag{5}$$

To completely specify our system we need two more equations; these are the equation for the drag on the sphere and the sphere's equation of motion. Now, if the drag is non-dimensionalized with $6\pi\mu R u_0$, we obtain

$$F_x = \frac{1}{3}\int_0^\pi \left\{\left[-p + 2\frac{\partial u_r}{\partial r}\right]\cos\theta - \left[\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right]\sin\theta\right\}_{r=1} \sin\theta\, d\theta. \tag{6}$$

Finally, the non-dimensional equation of motion of the sphere can be written as

$$F_x = (2/9\delta)(dU_p/dt). \tag{7}$$

Because of symmetry, the fluid velocity components can be derived in terms of a stream function ψ by means of

$$u_\theta = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}, \quad u_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}. \tag{8}$$

Equation (2) can also be written in terms of ψ by putting $\mathbf{u} = \nabla \times \mathbf{B}$, where $\mathbf{B} = (0, 0, \psi/r\sin\theta)$ is solenoidal. The result is

$$\left(E^2 - \frac{\partial}{\partial t}\right)E^2\psi = 0, \tag{9}$$

where

$$E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{10}$$

The boundary and initial conditions on ψ are

$$\partial\psi/\partial\theta = U_p \sin\theta\cos\theta, \quad \partial\psi/\partial r = U_p \sin^2\theta \quad \text{on} \quad r = 1, \tag{11}$$

$$\psi/r^2 \to \tfrac{1}{2}\sin^2\theta \quad \text{as} \quad r \to \infty \tag{12}$$

and

$$\psi(r, \theta, 0) = \psi_0 = \tfrac{1}{2}r^2\sin^2\theta(1 - 1/r^3). \tag{13}$$

Once ψ has been determined, u_r and u_θ can be obtained from (8). Similarly, the pressure can be obtained from

$$\frac{\partial p}{\partial\theta} = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(E^2\psi - \frac{\partial\psi}{\partial t}\right) \tag{14}$$

and

$$\frac{\partial p}{\partial r} = -\frac{1}{\sin\theta}\frac{\partial}{\partial r}\left(E^2\psi - \frac{\partial\psi}{\partial t}\right). \tag{15}$$

Because of the boundary and initial conditions, ψ is of the form

$$\psi = \sin^2\theta f(r,t). \tag{16}$$

The function $f(r,t)$ satisfies

$$\left(\frac{\partial^2}{\partial r^2}-\frac{2}{r^2}-\frac{\partial}{\partial t}\right)\left(\frac{\partial^2}{\partial r^2}-\frac{2}{r^2}\right)f(r,t) = 0, \tag{17}$$

together with the conditions

$$f(1,t) = \tfrac{1}{2}U_p, \quad \partial f(1,t)/\partial r = U_p, \tag{18}$$

$$f(r,t)/r^2 \to \tfrac{1}{2} \quad \text{for} \quad r\to\infty \tag{19}$$

and

$$f(r,0) \equiv f_0 = \tfrac{1}{2}(r^2 - 1/r). \tag{20}$$

2.2. *Transformed equations*

We now transform the above system using the transformation $f(r,t) \to \tilde{f}(r,s)$, where

$$\tilde{f}(r,s) = \int_0^\infty f(r,t)\, e^{-st}\, dt.$$

The differential equation for $\tilde{f}$ is, with $(d^2/dr^2 - 2/r^2)f_0 \equiv 0$,

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}-s\right)\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\tilde{f} = 0, \tag{21}$$

with

$$\tilde{f}(1,s) = \tfrac{1}{2}\tilde{U}_p, \quad d\tilde{f}(1,s)/dr = \tilde{U}_p \tag{22}$$

and

$$\tilde{f}(r,s) \to r^2/2s \quad \text{as} \quad r\to\infty. \tag{23}$$

The transformed drag is given by

$$\tilde{F}_x = -\frac{1}{3}\int_0^\pi \left[\tilde{p}\cos\theta + \sin\theta\frac{\partial \tilde{u}_\theta}{\partial r}\right]_{r=1} \sin\theta\, d\theta, \tag{24}$$

where we have used the fact that, on $r = 1$, $\partial\tilde{u}_r/\partial r = 0$ and $\partial\tilde{u}_r/\partial\theta = \tilde{u}_\theta$. The transformed equation of motion of the sphere is, with $U_p(0) = 0$,

$$\tilde{F}_x = (2s/9\delta)\,\tilde{U}_p, \tag{25}$$

and the transformed equations for the pressure are

$$\frac{\partial\tilde{p}}{\partial r} = \frac{2}{r^2}\cos\theta\left[\left(\frac{d^2}{dr^2}-\frac{2}{r^2}-s\right)\tilde{f}+f_0\right] \tag{26}$$

and

$$\frac{\partial\tilde{p}}{\partial\theta} = -\sin\theta\frac{d}{dr}\left[\left(\frac{d^2}{dr^2}-\frac{2}{r^2}-s\right)\tilde{f}+f_0\right]. \tag{27}$$

Now, the general solution of (21) is

$$\tilde{f} = \frac{A}{r} + Br^2 + C\left(\frac{r}{s^{\frac{1}{2}}}\right)^{\frac{1}{2}} K_{\frac{3}{2}}(s^{\frac{1}{2}}r) + D\left(\frac{r}{s^{\frac{1}{2}}}\right)^{\frac{1}{2}} I_{\frac{3}{2}}(s^{\frac{1}{2}}r),$$

where $(\pi/2Z)^{\frac{1}{2}} I_{\frac{3}{2}}(Z)$ and $(\pi/2Z)^{\frac{1}{2}} K_{\frac{3}{2}}(Z)$ are modified spherical Bessel functions of the first and third kind, respectively. This solution can be simplified because of the conditions at infinity, which impose $D = 0$ and $B = 1/2s$. Furthermore $(\pi/2Z)^{\frac{1}{2}} K_{\frac{3}{2}}(Z) = (\pi/2Z^2)(1+Z)\exp(-Z)$, so that

$$\tilde{f} = \frac{A}{r} + \frac{1}{2s} r^2 + \frac{C}{sr}(1+s^{\frac{1}{2}} r)\exp(-s^{\frac{1}{2}} r). \tag{28}$$

The constants A and C can be obtained from the conditions at $r = 1$ given by (22). Applying these conditions, we obtain

$$A = (\tilde{U}_p - 1/s)(s + 3s^{\frac{1}{2}} + 3)/2s^2 \tag{29}$$

and

$$C = -\tfrac{3}{2}(\tilde{U}_p - 1/s)\exp s^{\frac{1}{2}}. \tag{30}$$

The transformed sphere velocity $\tilde{U}_p$ can be obtained from the drag equation and from the sphere's dynamic equation of motion. We first solve for $\tilde{p}(1,\theta)$ from (20) and (26)–(28), the result being

$$\tilde{p}(1,\theta) = (sA + \tfrac{1}{2})\cos\theta + \text{constant}. \tag{31}$$

Similarly, we use $\tilde{u}_\theta = -(1/r)\sin\theta\,(d\tilde{f}/dr)$ to obtain

$$\frac{\partial u_\theta}{\partial r}(1,\theta) = -\sin\theta\left[2A + \frac{1}{s} + C\frac{s^{\frac{3}{2}} + s + 2s^{\frac{1}{2}} + 2}{s}\exp(-s^{\frac{1}{2}}) - \tilde{U}_p\right]. \tag{32}$$

Equations (31) and (32) are now inserted into (24). After integration this yields

$$\tilde{F}_x = \tfrac{4}{9}\left[2A + \frac{1}{s} + C\frac{s^{\frac{3}{2}} + s + 2s^{\frac{1}{2}} + 2}{s}\exp(-s^{\frac{1}{2}}) - \tilde{U}_p\right] - \tfrac{2}{9}(sA + \tfrac{1}{2}). \tag{33}$$

We now substitute the values of A and C from (29) and (30), respectively, and eliminate $\tilde{F}_x$ in favour of $\tilde{U}_p$ by means of (25). After some algebra we obtain

$$\tilde{U}_p = (9/s)(1+s^{\frac{1}{2}})\,[(2/\delta + 1)\,s + 9s^{\frac{1}{2}} + 9]^{-1}. \tag{34}$$

Finally, this value of $\tilde{U}_p$ and the above values of A and C yield

$$\tilde{f}(r,s) = \frac{1}{2rs}\frac{s + 3(s^{\frac{1}{2}}+1)}{s + \frac{9}{2}\delta(s^{\frac{1}{2}}+1)} + \frac{r^2}{2s} + \frac{3}{2sr}\frac{1 + s^{\frac{1}{2}} r}{s + \frac{9}{2}\delta(s^{\frac{1}{2}}+1)}e^{-s^{\frac{1}{2}}(r-1)}. \tag{35}$$

These equations can in principle be inverted to yield the sphere's velocity and, together with (16), the time-dependent stream function.

2.3. *Results*

Sphere's velocity. We are presently interested only in $U_p(t)$, especially for values of $\delta \leqslant 10^{-2}$. We thus neglect unity compared to $2/\delta$ in the denominator of $\tilde{U}_p$ and rewrite (34) as

$$\tilde{U}_p = \frac{9\delta}{2(\gamma-\beta)}\left\{\frac{1}{s(s^{\frac{1}{2}}-\gamma)} - \frac{1}{s(s^{\frac{1}{2}}-\beta)} + \frac{1}{s^{\frac{1}{2}}(s^{\frac{1}{2}}-\gamma)} - \frac{1}{s^{\frac{1}{2}}(s^{\frac{1}{2}}-\beta)}\right\}, \tag{36}$$

where

$$\gamma \approx -\tfrac{9}{4}\delta + 3i(\tfrac{1}{2}\delta)^{\frac{1}{2}}, \quad \beta \approx -\tfrac{9}{4}\delta - 3i(\tfrac{1}{2}\delta)^{\frac{1}{2}}. \tag{37}$$

Equation (36) is now inverted term by term using the inversion formulae (Roberts & Kaufman 1966):

$$L^{-1}\frac{1}{s(s^{\frac{1}{2}} + a - ib)} = \frac{1}{(a-ib)}\{1 - \exp[(a-ib)^2 t]\operatorname{erfc}[(a-ib)\,t^{\frac{1}{2}}]\} \tag{38}$$

and

$$L^{-1}\frac{1}{s^{\frac{1}{2}}(s^{\frac{1}{2}} + a - ib)} = \exp[(a-ib)^2 t]\operatorname{erfc}[(a-ib)\,t^{\frac{1}{2}}]. \tag{39}$$

The inverted sphere velocity $\bar{U}_p$ is thus given in terms of complementary error functions of complex argument, and is therefore not amenable to numerical computation. A more useful representation can be obtained by using the auxiliary function $W[Z(X+iY)]$ defined by (Abramowitz & Stegun 1964, p. 297)

$$W(Z) = \exp(-Z^2)\,\mathrm{erfc}\,(-iZ), \tag{40}$$

whose real and imaginary parts are tabulated for various values of X and Y. We therefore put $(a-ib)\,t^{\frac{1}{2}} = -iZ$ and $(a+ib)\,t^{\frac{1}{2}} = -i\zeta$, and find, for $\delta \ll 1$,

$$U_p(t) = 1 + (3/4i)\,(\tfrac{1}{2}\delta)^{\frac{1}{2}}\,[W(Z) - W(\zeta)] - \tfrac{1}{2}[W(Z) + W(\zeta)]. \tag{41}$$

Now, in view of the properties of $W(Z)$, and since $\zeta = -\bar{Z}$, where the overbar implies a complex conjugate, we can write $W(\zeta) = \overline{W(Z)}$. This enables us to write (41) as

$$U_p(t) = 1 + \tfrac{3}{2}(\tfrac{1}{2}\delta)^{\frac{1}{2}}\,\mathrm{Im}\,W(Z) - \mathrm{Re}\,W(Z), \tag{42}$$

where

$$Z = [1 + \tfrac{3}{2}i(\tfrac{1}{2}\delta)^{\frac{1}{2}}]\,T^{\frac{1}{2}}. \tag{43}$$

In the definition of Z we have put $T = \frac{9}{2}\delta t$. Physically, this quantity is the ratio of actual time to the relaxation time of the sphere, defined by (Fuchs 1964; Rudinger 1964)

$$\tau_d = 2R^2/9\nu\delta. \tag{44}$$

Equation (42) is the most important result of this paper; it gives the sphere velocity in terms of the tabulated functions Im $W(Z)$ and Re $W(Z)$ for any value of $Z = X + iY$, where $Y = \frac{3}{2}(\frac{1}{2}\delta)^{\frac{1}{2}} X$ and $X = T^{\frac{1}{2}}$. However, when δ is very small, we can obtain explicit expressions for Im $W(Z)$ and Re $W(Z)$ by expanding $W(Z)$ for small values of Y. Using the definition of $W(Z)$, we obtain

$$\mathrm{Re}\,W(Z) \approx \cos[3(\tfrac{1}{2}\delta)^{\frac{1}{2}}T]\,[\exp(-T) - 3(\delta/2\pi)^{\frac{1}{2}}T^{\frac{1}{2}}] + (2/\pi^{\frac{1}{2}})\,F(T^{\frac{1}{2}})\sin[3(\tfrac{1}{2}\delta)^{\frac{1}{2}}T] \tag{45}$$

and

$$\mathrm{Im}\,W(Z) \approx (2/\pi^{\frac{1}{2}})\,F(T^{\frac{1}{2}})\cos[3(\tfrac{1}{2}\delta)^{\frac{1}{2}}T] - \sin[3(\tfrac{1}{2}\delta)^{\frac{1}{2}}T] \times [\exp(-T) - 3(\delta/2\pi)^{\frac{1}{2}}T^{\frac{1}{2}}], \tag{46}$$

where

$$F(T) = \exp(-T^2)\int_0^T e^{u^2}\,du \tag{47}$$

is Dawson's integral.

It is of interest to compare our result for U_p with that obtained by assuming that the drag on the sphere is given by

$$F_x^{(S)} = 1 - U_p, \tag{48}$$

i.e. with $U_p^{(S)} = 1 - \exp(-T)$. Table 1 presents this comparison for various values of δ. It is seen that for these values of δ the difference is small, except for small T, and that, as δ decreases, the difference also diminishes. In fact, in the limit $\delta \to 0$ our result (42) reduces to that obtained using the Stokes drag. Physically, this means that, owing to the fact that its inertia is large compared with that of the surrounding gas, the sphere is at every instant in a quasi-steady Stokes flow. In other words, as the sphere accelerates from rest the fluid in its

$T^{\frac{1}{2}}$	U_p: $\delta = 10^{-2}$	$\delta = 10^{-3}$	$\delta = 10^{-4}$	$U_p^{(S)}$
0·02	0·00517	0·00191	0·00878	0·00040
0·04	0·01111	0·00461	0·00255	0·00160
0·06	0·01780	0·00810	0·00502	0·00360
0·08	0·02522	0·01237	0·00827	0·00640
0·10	0·03335	0·01739	0·01231	0·00996
0·20	0·08378	0·05348	0·04374	0·03922
0·40	0·22257	0·17200	0·15554	0·14786
0·60	0·38667	0·32970	0·31106	0·30233
0·80	0·54806	0·49707	0·48047	0·47271
1·00	0·68694	0·64958	0·63765	0·63213

TABLE 1. Numerical comparison of (42) with $U_p^{(S)} = 1-\exp(-T)$

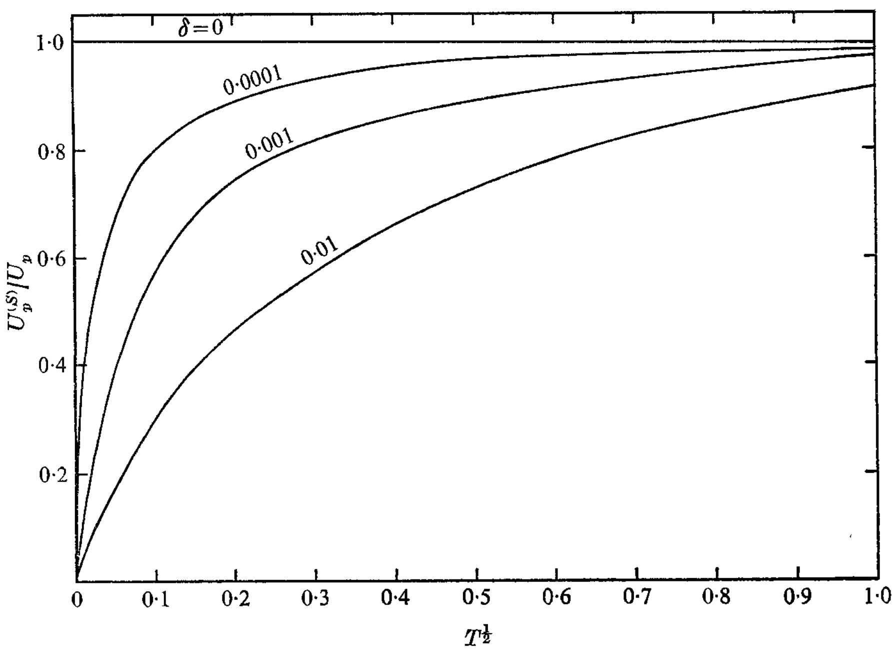

FIGURE 1. Ratio of the sphere velocity derived from Stokes's law to that predicted by (42).

vicinity is subjected to new boundary conditions, so that, in general, the sphere is surrounded by a flow field which is changing continuously. However, when the density of the sphere is large its acceleration is small, and the fluid in its vicinity adjusts to the new conditions in a time which is small compared with the time over which conditions change significantly. On the other hand, when δ is not infinitesimal, the differences are more significant, as is shown in table 1, or in figure 1, where the ratio $U_p^{(S)}/U_p$ is plotted versus $T^{\frac{1}{2}}$ for various values of δ.

Also of interest are the initial and asymptotic forms of (42). For $T \ll 1$, using (45) and (46), or the known series expansion for $W(Z)$, we obtain

$$U_p \approx 6(\delta/2\pi)^{\frac{1}{2}}T^{\frac{1}{2}}. \tag{49}$$

$T^{\frac{1}{2}}$	F_x $\delta = 10^{-2}$	$\delta = 10^{-3}$	$\delta = 10^{-4}$	$F_x^{(S)}$
0·02	6·97427	2·88895	1·59706	0·99960
0·04	3·97150	1·93855	1·29570	0·99840
0·06	2·96276	1·61817	1·19302	0·99640
0·08	2·45205	1·45472	1·13942	0·99358
0·10	2·14015	1·35360	1·10500	0·99004
0·20	1·46971	1·21190	1·01145	0·96078
0·40	0·99632	0·89597	0·86582	0·85214
0·60	0·70961	0·69854	0·69763	0·69767
0·80	0·48773	0·51142	0·52190	0·52729
1·00	0·31791	0·34910	0·36160	0·36787

TABLE 2. Numerical comparison of F_x from (51) with $F_x^{(S)} = \exp(-T)$

$T^{\frac{1}{2}}$	$X_p/u_0\tau_d$ $\delta = 10^{-2}$	$\delta = 10^{-3}$	$\delta = 10^{-4}$	$X_p^{(S)}/u_0\tau_d$
0·10	0·00022	0·00009	0·00006	0·00004
0·20	0·00182	0·00109	0·00090	0·00078
0·40	0·01978	0·01465	0·01300	0·01214
0·60	0·08024	0·06512	0·06019	0·05767
0·80	0·21124	0·18173	0·17211	0·16729
1·00	0·43591	0·38956	0·37506	0·36787

TABLE 3. Numerical comparison of $X_p/u_0\tau_d$ from (52) with $X_p^{(S)}/u_0\tau_d = T - U_p^{(S)}$

This equation can also be obtained, except for a factor of $\sqrt{2}$, from simple energy considerations of the initial flow over a flat plate, when the free-stream velocity is locally given by $\frac{3}{2}\sin\theta$. Finally, as $T \to \infty$, the asymptotic series for $W(Z)$ given by Abramowitz & Stegun (1964) yields (for small but finite δ)

$$U_p \approx 1 - \tfrac{3}{2}(\delta/2\pi)^{\frac{1}{2}}/T^{\frac{3}{2}}, \tag{50}$$

instead of the exponential limit predicted by Stokes's law.

Displacement and drag. For completeness, we present results for the sphere's displacement and drag. The drag is found directly from (41) together with $\zeta^2 W(\zeta) - \bar{Z}^2\overline{W(Z)}$, the result being

$$F_x = 3(\delta/2\pi T)^{\frac{1}{2}} + (1/T)\,\mathrm{Re}\,[Z^2W(Z)] - (3/2T)\,(\tfrac{1}{2}\delta)^{\frac{1}{2}}\,\mathrm{Im}\,[Z^2W(Z)]. \tag{51}$$

The displacement is obtained from (42) by integration, the function $W(Z)$ being integrated as follows:

$$I = \int_0^T W[Z(T')]\,dT' = \int_0^T \exp\left[-(1+id)^2T'\right]\left[1+\frac{2i}{\sqrt{\pi}}\int_0^{(1+id)T'^{\frac{1}{2}}} \exp u^2\,du\right] dT',$$

where $d = \frac{3}{2}(\frac{1}{2}\delta)^{\frac{1}{2}}$. Interchanging the order of integration in the second term, we obtain

$$I = [1 + 2iZ/\sqrt{\pi} - W(Z)]/(1+id)^2.$$

The final result for the sphere's displacement is

$$X_p/u_0\tau_d = T - (1+\tfrac{9}{8}\delta)^{-1}\,[1 - \mathrm{Re}\,W(Z) - \tfrac{3}{2}(\tfrac{1}{2}\delta)^{\frac{1}{2}}\,\mathrm{Im}\,W(Z)]. \tag{52}$$

Tables 2 and 3 compare some values of F_x and $X_p/u_0\tau_d$ as predicted by (51) and (52) respectively, with $F_x^{(S)}$ and $X_p^{(S)}/u_0\tau_d$. Here $X_p^{(S)}/u_0\tau_d = T-1+\exp(-T)$ is the non-dimensional displacement obtained using the instantaneous Stokes drag $F_x^{(S)}$.

3. Discussion

The response of a rigid sphere to an acoustic pulse has been studied by means of an approximate model. The model neglects diffraction of the pulse front, and ignores finite Reynolds number effects in dealing with the transient problem. The latter effects are probably quite significant, as the work of Ockendon (1968) shows. Nevertheless, our results should provide a first approximation for the sphere displacement in the small Reynolds number limit. Our results also show that, when δ is not infinitesimal, sphere trajectories differ considerably, especially for small times, from those computed using the Stokes drag. This result is of some importance in the study of shock-induced droplet collisions in aerosol clouds. Finally, for $\delta \to 0$ our results reduce to those obtained using the Stokes drag. This limiting behaviour had been noticed previously (Temkin & Dobbins 1966) and provides further support for the use of the Stokes drag in some unsteady, small Reynolds number flows of dusty gases.

The author is grateful to Dr D. G. Briggs for some helpful discussions, and to Mr J. M. Reichman for his help with the numerical computations. This work was partially supported by the National Science Foundation.

Appendix. Inviscid solution

We consider a unit-step acoustic pulse travelling in a dispersionless medium with a speed a and reaching the edge of a sphere at $t = 0$. The resulting motion can be described by the sum ϕ of an incident potential ϕ_i and a scattered wave potential ϕ_s. If the sphere's centre is initially at $x = 0$ and the pulse is travelling along the x axis, ϕ_i is given by

$$\phi_i(x,t) = x+1-ct, \tag{A 1}$$

with

$$\phi_i(x) = 0 \quad \text{for} \quad x \geqslant -1, \tag{A 2}$$

where $c = aR/\nu$. The scattered potential is given by the solution of

$$\partial^2\phi_s/\partial t^2 - c^2\nabla^2\phi_s = 0 \tag{A 3}$$

and must, together with ϕ_i, satisfy the conditions $\partial\phi/\partial r = U_p\cos\theta$ on $r = 1$, $\nabla\phi \to \mathbf{e}_1$ as $r\to\infty$, and $U_p = 0$ for $t \leqslant 0$. The Laplace transforms of (A 1) and (A 3) are

$$\tilde{\phi}_i = A(s)\exp(-Kx) \tag{A 4}$$

and

$$\nabla^2\tilde{\phi}_s = K^2\tilde{\phi}_s, \tag{A 5}$$

where $A(s) = -(c/s^2)\exp(-K)$ and $K = s/c$. The solution of (A 5) can be written as

$$\tilde{\phi}_s = \sum_{n=0}^{\infty} i^n(2n+1)\,C_n h_n^{(1)}(iKr)\,P_n(\cos\theta), \tag{A 6}$$

where $h_n^{(1)}$ is the spherical Bessel function of the third kind and $P_n(\cos\theta)$ is the Legendre polynomial of order n. Similarly, the transformed incident potential has an expansion of the form

$$\tilde{\phi}_i = A(s) \sum_{n=0}^{\infty} i^n(2n+1)\, j_n(iKr)\, P_n(\cos\theta), \tag{A 7}$$

where j_n is the spherical Bessel function of the first kind. Because of the boundary conditions, we find that the only non-zero term is that with $n = 1$, so that (dropping the superscript on $h_1^{(1)}$)

$$\tilde{\phi} = 3i[Aj_1(iKr) + C_1 h_1(iKr)]\cos\theta. \tag{A 8}$$

Now, in terms of ϕ, the pressure is given by $p = -\partial\phi/\partial t$, so that $\tilde{p} = -cK\tilde{\phi}$ and the transformed pressure drag is

$$\tilde{F}_x = \tfrac{4}{3}iKc[Aj_1(iK) + C_1 h_1(iK)]. \tag{A 9}$$

By using (A 8) together with the boundary conditions on $r = 1$ we obtain

$$\tilde{U}_p = -3K[Aj_1'(iK) + C_1 h_1'(iK)], \tag{A 10}$$

where the primes denote differentiation with respect to the argument. The constant C_1 is obtained from (A 9) and (A 10) by using $\tilde{F}_x = (2cK/9\delta)\tilde{U}_p$:

$$C_1 = -A(s)\frac{j_1(iK)\,\delta - iKj_1'(iK)}{h_1(iK)\,\delta - iKh_1'(iK)}. \tag{A 11}$$

These results can be greatly simplified because the Bessel functions appearing in (A 11) can be expressed in terms of elementary functions (see, for example Abramowitz & Stegun 1964, chap. 10). We substitute the explicit values of j_1, h_1, j' and h_1' into (A 11) and (A 9) and obtain, after a lengthy but straightforward calculation,

$$\tilde{F}_x = \tfrac{2}{3}[2 + \delta + (2+\delta)\,K + K^2]^{-1}. \tag{A 12}$$

We invert this result for δ small and obtain

$$F_x = \tfrac{2}{3}c\sin(ct)\exp(-ct). \tag{A 13}$$

Similarly, the sphere velocity U_p and fluid tangential velocity at $r = 1$ are found to be

$$U_p(t) = \tfrac{3}{2}\delta\{1 - \exp(-ct)\,[\sin(ct) + \cos(ct)]\} \tag{A 14}$$

and

$$u_\theta(1,\theta,t) = -\tfrac{3}{2}\sin\theta\{1 - \exp(-ct)\,[\sin(ct) + \cos(ct)]\}. \tag{A 15}$$

For $t \to \infty$ these yield the classical results

$$U_p = \tfrac{3}{2}\delta, \quad u_\theta(1,\theta) = -\tfrac{3}{2}\sin\theta. \tag{A 16}$$

However, these values are reached rather rapidly. For instance, when $ct = 5$, where $ct = at'/R$ and t' is the real time, $|u_\theta| > 0{\cdot}99(3\sin\tfrac{1}{2}\theta)$. Since our viscous model assumes (A 16) as the initial conditions, it appears that its validity is limited to times of the order of $5R/a$ or larger. In terms of the variable T in (42), this criterion imposes the condition $T \geqslant (5R/a\tau_d)$. For a $20\,\mu$m sphere in air, for example, our results would then be limited to values of $T > 10^{-4}$. For times less than $5R/a$ it is necessary to consider the transient scattering of an acoustic pulse in a viscous gas. This can be done by means of a method entirely similar to the one used in this appendix, but taking into account transversely scattered waves of viscous origin.

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